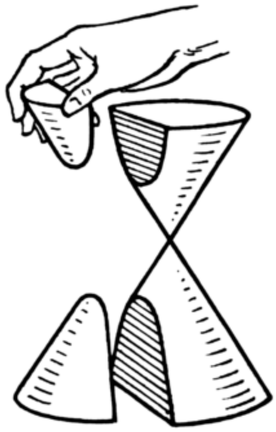
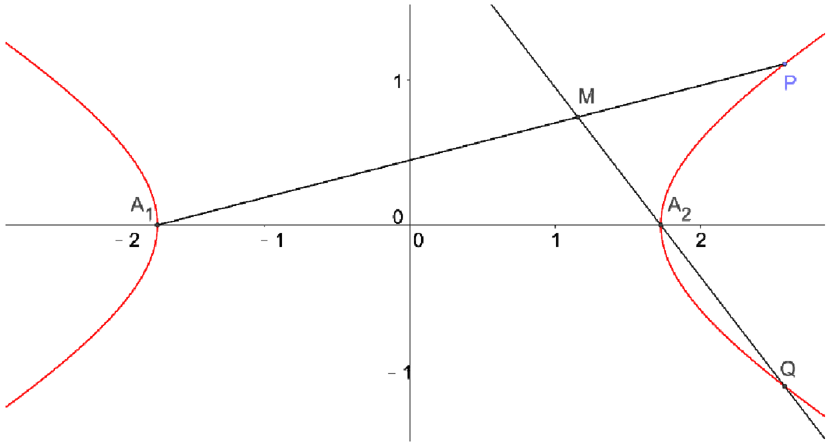
A hyperbola locus problem



Let the left and right vertices of the hyperbola $H\colon \frac{x^2}{3} - y^2 = 1$ be A_1, A_2.

Let $P(x_1, y_1),\ Q(x_1, -y_1)$ be two different movable points on the given hyperbola.

Let the intersection of A_1P and A_2Q be M. Find the equation E of the locus of M.



Method 1

From the question $|x_1| > 3$ and $A_1(-\sqrt{3},\ 0), A_2(\sqrt{3},\ 0)$.

$$PA_1\colon\ y = \frac{y_1}{x_1 + \sqrt{3}}(x + \sqrt{3}) \quad \ldots.(1)$$

$$QA_2\colon\ y = \frac{-y_1}{x_1 - \sqrt{3}}(x - \sqrt{3}) \quad \ldots.(2)$$

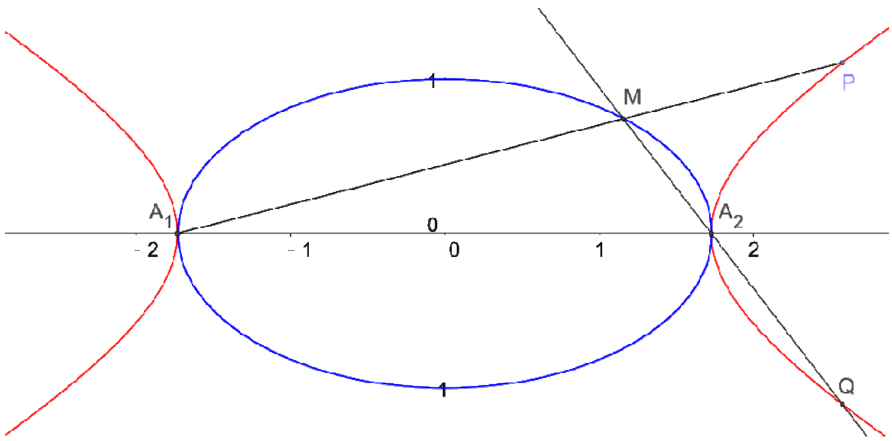
Solving, we get: $x = \frac{3}{x_1},\ y = \frac{\sqrt{3}y_1}{x_1}$

Change of subjects, we get: $x_1 = \frac{3}{x},\ y_1 = \frac{\sqrt{3}y}{x} \quad \ldots.(3)$

Then $x \neq 0,\ |x| < 3$.

Since $P(x_1, y_1)$ is on the hyperbola : $\frac{x^2}{3} - y^2 = 1$, we have $\frac{{x_1}^2}{3} - {y_1}^2 = 1 \quad \ldots.(4)$

$$(3) \downarrow (4),\quad E\colon\ \frac{x^2}{3} + y^2 = 1,\ x \neq 0,\quad x \neq \pm\sqrt{3}$$



In the above, $x \neq 0,\ x \neq \pm\sqrt{3}$ should be included in E. We can also get this by another way.

Assertion: $x \neq 0,\ x \neq \pm\sqrt{3}$ in E.

P, Q are two different points on the given hyperbola, so they cannot coincide with A_1 and A_2.

Therefore A_1 and A_2 cannot be points in E and $x \neq \pm\sqrt{3}$

Let L be the line joining $(0,1)$ and $A_2(\sqrt{3},\ 0)$.

Solving : $\begin{cases} L:\ x + \sqrt{3}y - \sqrt{3} = 0 \\ H:\ \frac{x^2}{3} - y^2 = 1 \end{cases}$

We get $x = \sqrt{3},\ y = 0$, Hence the only intersection point between L and H is $A_2(\sqrt{3},\ 0)$.

Hence E cannot pass through $(0,1)$.

Similarly E cannot pass through $(0,-1)$.

Method 2

From the question $|x_1| > 3$ and $A_1(-\sqrt{3},\ 0), A_2(\sqrt{3},\ 0)$.

Let $M(x,y)$ be the intersection point of A_1P and A_2Q .

A_1, M, P are collinear, $\frac{y}{x+\sqrt{3}} = \frac{y_1}{x_1+\sqrt{3}}$ ....(5)

A_2, M, Q are collinear, $\frac{y}{x-\sqrt{3}} = \frac{-y_1}{x_1-\sqrt{3}}$ ....(6)

$(5) \times (6)$, $\frac{y^2}{x^2-3} = \frac{-y_1{}^2}{x_1{}^2-3}$ ....(7)

Since $P(x_1, y_1)$ is on the hyperbola : $\frac{x^2}{3} - y^2 = 1$, we have $\frac{x_1{}^2}{3} - y_1{}^2 = 1.$

$$-y_1{}^2 = 1 - \frac{x_1{}^2}{3} \quad ....(8)$$

$(8) \downarrow (7)$, $E:\ \frac{x^2}{3} + y^2 = 1$

By the **Assertion** , $x \neq 0,\ x \neq \pm\sqrt{3}$.

Further point of interest for you to try :

Let the left and right vertices of the ellipse $E:\ \frac{x^2}{3} - y^2 = 1$ be A_1, A_2.

Let $P(x_1, y_1),\ Q(x_1, -y_1)$ be two different movable points on the given ellipse.

Let the intersection of A_1P and A_2Q be M. Find the equation H of the locus of M.